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Orderings of N-Tuples

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SPECIAL CASE

PROOF

We begin with the special case of n -tuples which sum to a given value, k , and build up to the general case. Let \mathbf{Z}^+ denote the positive integers and let $\mathbf{S}(n, k) = \{\alpha = (a_1, \dots, a_n), a_i \in \mathbf{Z}^+ \text{ s.t. } \sum a_i = k\}$ ordered lexicographically. It is well known that $|\mathbf{S}(n, k)|$ is simply the binomial coefficient $\binom{k-1}{n-1}$ choose $(n-1)$, i.e.,

$$|\mathbf{S}(n, k)| = \binom{k-1}{n-1}. \quad (1)$$

Now given an $\alpha \in \mathbf{S}(n, k)$, can we calculate its position in the ordering? Certainly $(1, 1, \dots, k+1-n)$ is first and $(k+1-n, 1, \dots, 1)$ is last. Let the order function on $\mathbf{S}(n, k)$ be $f_{n,k}: \mathbf{S}(n, k) \rightarrow \{1, 2, \dots, |\mathbf{S}(n, k)|\}$.

LEMMA 1.1

Let $\mathbf{S}(n, k)$ be the set of n -tuples of positive integers which sum to k , ordered lexicographically. Let $\alpha = (a_1, \dots, a_n)$ and define $\sigma_j(\alpha) = \sum_{i=j}^n a_i$. Then the position of α is given by

$$f_{n,k}(\alpha) = \binom{k-1}{n-1} - \sum_{j=1}^{n-1} \left(\sigma_{j+1} + \binom{a_j}{n-j} - 1 \right). \quad (2)$$

PROOF

Consider a fixed $\alpha = (a_1, \dots, a_n)$ in $\mathbf{S}(n, k)$. Let $\beta = (b_1, \dots, b_n)$, $T_j = \{\beta \in \mathbf{S}(n, k) \mid b_1 = a_1, \dots, b_{j-1} = a_{j-1}, b_j > a_j\}$ and let $T = \bigcup_{j=1}^n T_j$, $j = 1, \dots, n$. Then certainly $T = \{\beta \in \mathbf{S}(n, k) \mid \beta > \alpha\}$ and $T_i \cap T_j = \emptyset$ if $i \neq j$, so we have $|T| = \sum_{1 \leq j \leq n} |T_j|$ (since $T_n = \emptyset$). But $\{(T_j = (a_1, \dots, a_{j-1}, a_j + g_1, g_2, \dots, g_{n-j+1}) \mid g = (g_1, \dots, g_{n-j+1}) \in \mathbf{S}(n-j+1, k - a_1 - \dots - a_j)\}$. Hence,

$$|T_j| = \binom{k - a_1 - \dots - a_j - 1}{n-j} = \binom{\sigma_{j+1} + \binom{a_j}{n-j} - 1}{n-j}. \quad (3)$$

Therefore, we can conclude that $f_{n,k}(\alpha) = |\mathbf{S}(n, k)| - |\{b \in \mathbf{S}(n, k) \text{ s.t. } \beta > \alpha\}|$

$$= \binom{k-1}{n-1} - |T| = \binom{k-1}{n-1} - \sum_{j=1}^{n-1} |T_j| = \binom{k-1}{n-1} - \sum_{j=1}^{n-1} \left(\sigma_{j+1} + \binom{a_j}{n-j} - 1 \right).$$

qed

N-TUPLES OF POSITIVE INTEGERS

Now let $\mathbf{Z}^n = \mathbf{Z}^+ \times \dots \times \mathbf{Z}^+$ be the set of all n-tuples of positive integers. Note that $\mathbf{Z}^n = \bigcup_{n \leq k} \mathbf{Z}^n$. Let \mathbf{Z}^n be ordered by $S(n, n) < S(n, n+1) < S(n, n+2) < \dots$ where the $S(n, k)$ are ordered lexicographically as before. Let $f_n : \mathbf{Z}^n \rightarrow \mathbf{Z}$ be the order function for this space.

THEOREM 1.1

Let \mathbf{Z}^n be the set of all n-tuples of positive integers ordered as above. Let $\alpha = (a_1, \dots, a_n) \in \mathbf{Z}^n$ and define $\sigma_j(\alpha) = \sum_{i=j}^n a_i$. The position of α is given by

$$f_n(\alpha) = \binom{\sigma_j(\alpha)}{n} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha) - 1}{n-j}. \quad (4)$$

PROOF

By ordering on \mathbf{Z}^n and the lemma we have

$$\begin{aligned} f_n(\alpha) &= \sum_{j=1}^{\sigma_1(\alpha)-1} |S(n, k)| + f_{n, \sigma_1(\alpha)}(\alpha) = \sum_{j=1}^{\sigma_1(\alpha)-1} \binom{j-1}{n-1} + f_{n, \sigma_1(\alpha)}(\alpha) \\ &= \binom{\sigma_1(\alpha)-1}{n} + f_{n, \sigma_1(\alpha)}(\alpha) = \binom{\sigma_1(\alpha)-1}{n} + \binom{\sigma_1(\alpha)-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha)-1}{n-j} \\ &= \binom{\sigma_1(\alpha)}{n} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha)-1}{n-j} \left[\begin{array}{l} \text{using the basic combinatorial identity} \\ \binom{a+1}{b} = \binom{a}{b} + \binom{a}{b-1} \end{array} \right]. \end{aligned}$$

qed

COROLLARIES

Next I give two corollaries of lemma 1.1.

COROLLARY 3.1

Let $S(k) = \bigcup_{1 \leq n \leq k} S(n, k)$ ordered by $S(1, k) < \dots < S(k, k)$ where the $S(n, k)$ are ordered lexicographically as before. Then the lexicographic order function, $f^{(k)}$, of $\alpha = (a_1, \dots, a_n) \in S(k)$ is given by

$$f^{(k)}(\alpha) = \sum_{\lambda=1}^n \binom{k-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha) - 1}{n-j}. \quad (5)$$

PROOF

Using lemma 1.1, we have

$$\begin{aligned} f^{(k)}(\alpha) &= \sum_{\lambda=1}^{n-1} |S(n, k)| + f_{n,k}(\alpha) = \sum_{\lambda=1}^{n-1} \binom{k-1}{\lambda-1} + \binom{k-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha) - 1}{n-j} \\ &= \sum_{\lambda=1}^n \binom{k-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha) - 1}{n-j} \end{aligned}$$

qed

COROLLARY 2

Let $Z^* = \bigcup_{1 \leq n < \infty} Z^n = \{\alpha = (a_1, \dots, a_n), a_i \in \mathbb{Z}^+\}$ ordered by $S(1) < S(2) < \dots$ with the $S(k)$ defined and ordered as above. Then the order of function, f^* , of $\alpha = (a_1, \dots, a_n) \in Z^*$ is given by

$$f^*(\alpha) = 2^{\sigma_1(\alpha)-1} + \sum_{\lambda=1}^n \binom{\sigma_1(\alpha) - 1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha) - 1}{n-j}. \quad (6)$$

PROOF

Since $|S(m)| = \sum_{j=1}^m |S(j, m)| = \sum_{j=1}^m \binom{m-1}{j-1} = 2^{(m-1)}$, we have, by the ordering on Z^* ,

$$\begin{aligned} f^*(a) &= \sum_{\lambda=1}^{\sigma_1(a)-1} |S(m)| + f^{\sigma_1(a)-1}(a) = \sum_{m=1}^{\sigma_1(a)-1} 2^{m-1} + \sum_{\lambda=1}^n \binom{\sigma_1(a)-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j} \\ &= 2^{\sigma_1(a)-1} + \sum_{\lambda=1}^n \binom{\sigma_1(a)-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j} . \end{aligned}$$

qed

INVERSE FUNCTION

Let us consider the inverse function, $f_n^{-1}: Z \rightarrow Z^n$. There does not appear to be a closed form solution, but it is readily computable. Using this, one can easily implement an algorithm to calculate

$\phi_{k,n}: Z^k \rightarrow Z^n$ by $\phi_{k,n} = f_n^{-1} f_k$. Now suppose p in Z^+ . To compute $f_n^{-1}(p) = (a_1, \dots, a_n)$ first find the

smallest k_1 s.t. $\binom{k_1}{n} \geq p$. Then we find successively largest k_i , $i = 2, \dots, n$, s.t. $\binom{k_i}{n} - \sum_{j=1}^{i-1} \binom{k_{j+1}}{n-j} \geq p$.

For the n^{th} case, this expression will be an equality. Then $k_i = \sigma_i(f_n^{-1}(p))$ for $i = 1, \dots, n$ and given all the σ s, one easily finds (a_1, \dots, a_n) .

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